

COLOURED QUIVERS OF TYPE A AND THE CELL-GROWTH PROBLEM

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ABSTRACT. We use a geometric description of m -cluster categories of Dynkin type A to count the number of coloured quivers in the m -mutation class of quivers of Dynkin type A . This is related to angulations of polygons and the cell-growth problem.

INTRODUCTION

Quiver mutation [FZ1] induces an equivalence relation on the set of quivers, and the mutation class of a quiver Q consists of all quivers mutation equivalent to Q . It was shown in [BR] that the mutation class of an acyclic quiver Q is finite if and only if the underlying graph of Q is either Dynkin, extended Dynkin or has at most two vertices. In [To2] an analogous result was obtained for coloured quivers. Let Q be a coloured quiver obtained from an m -cluster tilting object. Then Q has finite mutation class if and only if Q is mutation equivalent to a quiver Q' , where the Gabriel quiver of Q' is Dynkin, extended Dynkin or has at most two vertices, and Q' has only arrows of colour 0 and m .

For finite mutation classes, it is a natural question to ask how many non-isomorphic quivers there are in the class. In [To1, BTo, BRS] explicit formulas were given to compute the number of non-isomorphic quivers in the mutation class of quivers of type A , D and \tilde{A} respectively, and in this paper we will give an explicit formula for the number of non-isomorphic coloured quivers of type A . This generalizes a result in [To1].

In [BM1, BM2] the authors gave geometric descriptions of the m -cluster categories of type D and A . This generalizes the geometric descriptions of cluster categories given in [CCS, S]. In this paper we will use this to investigate mutation classes arising from m -cluster categories of type A .

There is a $1 - 1$ correspondence between m -cluster tilting objects and $(m + 2)$ -angulations of an $(nm + 2)$ -gon in this case. This enables us to count the number of m -cluster tilting objects in the m -cluster category. Furthermore we show that there is a $1 - 1$ correspondence between coloured quivers of m -cluster tilted algebras and $(m + 2)$ -angulations of $(nm + 2)$ -gons, where two $(m + 2)$ -angulations are considered equivalent if they are rotations of each other. Counting such $(m + 2)$ -angulations is related to the cell-growth problem investigated by for example Harary, Palmer and Read in [HPR], and they provide the generating functions that we need.

1. m -CLUSTER CATEGORIES

Cluster categories were defined in [BMRR] in the general case and in [CCS] in the A -case as a categorical model of the combinatorics of cluster algebras. Some cluster categories have a nice geometric description in terms of triangulations of certain polygons, see [CCS, S].

Let $H = kQ$ be a finite dimensional hereditary algebra over an algebraically closed field k , where Q is a quiver with n vertices. Let $\mathcal{D}^b(H)$ be the bounded derived category of $\text{mod } H$. The orbit category $\mathcal{C}_H = \mathcal{D}^b(H)/\tau^{-1}[1]$, where τ is

the Auslander-Reiten translation and $[1]$ is the shift functor, is called the cluster category of H . In [CCS] this category was defined as a category of diagonals of a regular $(n+3)$ -gon. The objects are direct sums of diagonals and the morphisms are spanned by elementary moves modulo the mesh-relations. For the D case [S] considered n -gons with a puncture. Schiffler defined a category with direct sums of diagonals (or tagged edges) as objects. The morphism space is spanned by elementary moves modulo the mesh relations. He showed that this category was equivalent to the cluster category of type D .

A generalization of cluster categories are the m -cluster categories. Consider the orbit category $\mathcal{C}_H^m = \mathcal{D}^b(H)/\tau^{-1}[m]$, for some positive integer m . This category is called the m -cluster category of H . It has been investigated in several papers, see for example [BM1, BM2, BT, IY, K, T, W, Z, ZZ]. The m -cluster category is a Krull-Schmidt category for all m , and it has an AR-translate τ . From [K] we also know that it is a triangulated category for all m . The indecomposable objects in \mathcal{C}_H^m are of the form $X[i]$, with $0 \leq i < m$, where X is an indecomposable H -module, or of the form $P[m]$, where P is a projective H -module.

Baur and Marsh considered in [BM1, BM2] m -cluster categories, and they generalized [S, CCS] for arbitrary m 's. In this paper we will consider their generalization of geometric descriptions of m -cluster categories of type A .

If T is an object in \mathcal{C}_H^m with the property that X is in $\text{add } T$ if and only if $\text{Ext}_{\mathcal{C}_H^m}^i(T, X) = 0$ for all $i \in \{1, 2, \dots, m\}$, then T is called an m -cluster tilting object. An object X is called maximal m -rigid if it has the property that $X \in \text{add } T$ if and only if $\text{Ext}_{\mathcal{C}_H^m}^i(T \oplus X, T \oplus X) = 0$ for all $i \in \{1, 2, \dots, m\}$. In [W, ZZ] it was shown that an object which is maximal m -rigid is also an m -cluster tilting object, and that an m -cluster tilting object T always has n non-isomorphic indecomposable summands [Z].

If T is an m -cluster tilting object in \mathcal{C}_H^m , the algebra $\text{End}_{\mathcal{C}_H^m}(T)$ is called an m -cluster tilted algebra. In [To1] it was shown that 1-cluster tilted algebras of type A_n (up to isomorphism) are in 1-1 correspondence with triangulations of $(n+3)$ -gons, where two triangulations are considered equivalent if they are rotations of each other. A similar result was obtained in [BTo] for type D . We will see that this does not hold for an arbitrary m .

2. QUIVER MUTATION

Let \bar{T} be an object in \mathcal{C}_H^m with $n-1$ non-isomorphic indecomposable direct summands such that $\text{Ext}_{\mathcal{C}_H^m}^i(\bar{T}, \bar{T}) = 0$ for $i \in \{1, 2, \dots, m\}$. Such an object is called an almost complete m -cluster tilting object, and in [W, ZZ] they show that \bar{T} always has exactly $m+1$ complements.

Let $T_k^{(c)}$, where $c \in \{0, 1, 2, \dots, m\}$, be the complements of $\bar{T} = T/T_k$. Then the complements are connected by $m+1$ exchange triangles [IY]

$$T_k^{(c)} \rightarrow B_k^{(c)} \rightarrow T_k^{(c+1)} \rightarrow,$$

where $B_k^{(c)}$ is in $\text{add } \bar{T}$.

Let T be an m -cluster tilting object in \mathcal{C}_H^m . In [BT] the authors associate to T a quiver Q_T in the following way. There is a vertex in Q_T for every indecomposable summand of T , and the arrows have colours chosen from the set $\{0, 1, 2, \dots, m\}$. If T_i and T_j are two indecomposable summands of T corresponding to vertex i and j in Q_T , there are r arrows from i to j of colour c , where r is the multiplicity of T_j in $B_i^{(c)}$.

They show that quivers obtained in this way have the following properties.

- (1) There are no loops.

- (2) If there is an arrow from i to j with colour c , then there exists no arrow from i to j with colour $c' \neq c$.
- (3) If there are r arrows from i to j of colour c , then there are r arrows from j to i of colour $m - c$.

Coloured quiver mutation keeps track of the exchange of indecomposable summands of an m -cluster tilting object. The mutation of Q_T at vertex j is a quiver $\mu_j(Q_T)$ obtained as follows.

- (1) For each pair of arrows

$$i \xrightarrow{(c)} j \xrightarrow{(0)} k$$

where $i \neq k$ and $c \in \{0, 1, \dots, m\}$, add an arrow from i to k of colour c and an arrow from k to i of colour $m - c$.

- (2) If there exist arrows of different colours from a vertex i to a vertex k , cancel the same number of arrows of each colour until there are only arrows of the same colour from i to k .
- (3) Add one to the colour of all arrows that goes into j , and subtract one from the colour of all arrows going out of j .

In [BT] the following theorem is proved.

Theorem 2.1. *Let $T = \oplus_{i=1}^n T_i$ be an m -cluster tilting object in \mathcal{C}_H^m . Let $T' = T/T_j \oplus T_j^{(1)}$ be an m -cluster tilting object where there is an exchange triangle*

$$T_j \rightarrow B_j^{(0)} \rightarrow T_j^{(1)} \rightarrow .$$

Then $Q_{T'} = \mu_j(Q_T)$.

Let us denote by Q_G , where $Q = Q_T$ is a coloured quiver and T an m -cluster tilting object, the quiver obtained from Q by removing all arrows with colour different from 0. Then the m -cluster tilted algebra $\text{End}_{\mathcal{C}_H^m}(T)$ has Gabriel quiver Q_G . From [ZZ] we have the following proposition and corollary. See also [BT].

Proposition 2.2. *Any m -cluster tilting object can be reached from any other m -cluster tilting object via iterated mutation.*

Corollary 2.3. *For an m -cluster category \mathcal{C}_H^m of the acyclic quiver Q , all quivers of m -cluster tilted algebras are given by repeated mutation of Q .*

For $m = 1$, it was shown in [BR] that the mutation class of an acyclic quiver Q is finite if and only if the underlying graph of Q is either Dynkin, extended Dynkin or Q has at most two vertices. This was generalized in [To2], and we have the following theorem and corollary.

Theorem 2.4. *Let k be an algebraically closed field and Q a connected finite quiver without oriented cycles. The following are equivalent for $H = kQ$.*

- (1) *There are only a finite number of basic m -cluster tilted algebras associated with H , up to isomorphism.*
- (2) *There are only a finite number of Gabriel quivers occurring for m -cluster tilted algebras associated with H , up to isomorphism.*
- (3) *H is of finite or tame representation type, or has at most two non-isomorphic simple modules.*
- (4) *There are only a finite number of τ -orbits of m -cluster tilting objects associated with H .*
- (5) *There are only a finite number of coloured quivers occurring for m -cluster tilting objects associated with H , up to isomorphism.*
- (6) *The mutation class of a coloured quiver arising from an m -cluster tilting object associated with H , is finite.*

Corollary 2.5. *A coloured quiver Q corresponding to an m -cluster tilting object, has finite mutation class if and only if Q is mutation equivalent to a quiver Q' , where Q'_G has underlying graph Dynkin or extended Dynkin, or it has at most two vertices, and there are only arrows of colour 0 and m in Q' .*

3. CATEGORY OF m -DIAGONALS OF A REGULAR $(nm + 2)$ -GON

From now on we want to consider only the A case. Baur and Marsh gave in [BM2] a geometric description of the m -cluster category in the A case, and we will use this to investigate the mutation class of coloured quivers.

Let n and m be positive integers and $P_{n,m}$ a regular $(nm + 2)$ -gon. Label the vertices on the border of the polygon clockwise from 1 to $nm + 2$. See Figure 1.

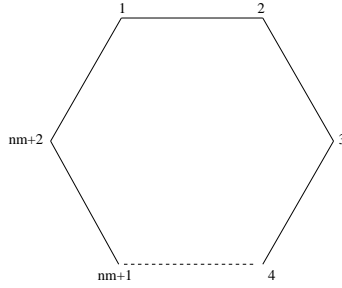


FIGURE 1. The $(nm + 2)$ -gon $P_{n,m}$.

An m -diagonal α is a straight line between two vertices on the border such that α divides the polygon into two parts, each with number of sides congruent to 2 modulo m . We denote by $\alpha_{i,j}$ the m -diagonal between the vertex i and j .

We say that a set Δ of m -diagonals do not cross if no two diagonals in Δ cross at the interior of the polygon. A maximal set of m -diagonals that do not cross consists of exactly $n - 1$ diagonals, and they divide the polygon into $(m + 2)$ -gons. We call such a set an $(m + 2)$ -angulation of $P_{n,m}$. See [BM2]. We note that if $m = 1$, a maximal set of diagonals is a triangulation of the regular polygon $P_{n,1}$, and it consists of $n - 1$ diagonals. See Figure 2 for some examples.

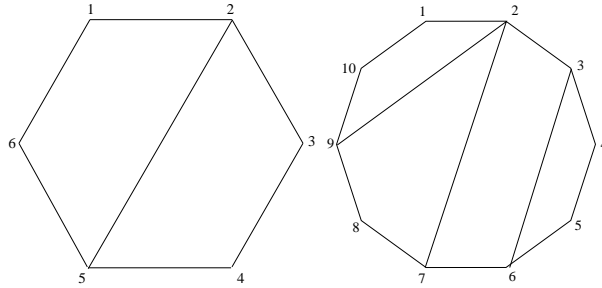
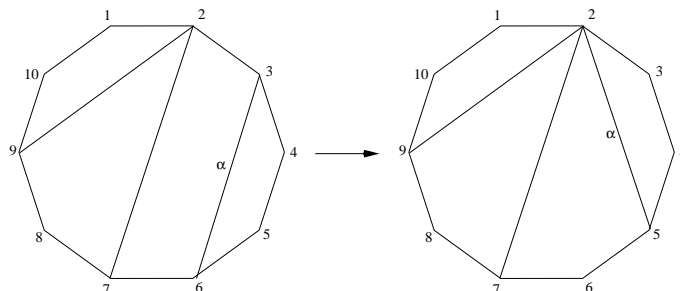


FIGURE 2. A 4-angulation of $P_{2,2}$ and a 4-angulation of $P_{4,2}$.

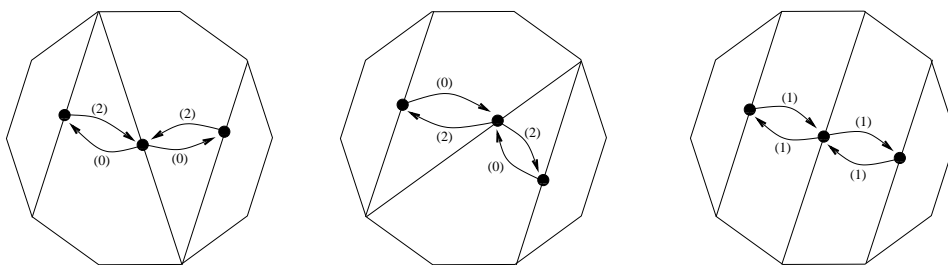
Given an $(m+2)$ -angulation Δ of $P_{n,m}$, [BT] define an operation on the diagonals. Let α be an m -diagonal in Δ , and consider the set of diagonals $(\Delta \setminus \alpha)$. By removing α , we obtain a $(2m + 2)$ -gon, and there are $m + 1$ possibilities to put it back to obtain a new $(m + 2)$ -angulation. We call α a diameter of the inner $(2m + 2)$ -gon, because it geometrically connects two opposite vertices in the $(2m + 2)$ -gon. If we

replace α with another diagonal, say β , then β is also a diameter of the $(2m+2)$ -gon, and hence all possibilities to replace α with another diagonal correspond to rotating the $(2m+2)$ -gon. The operation is defined as follows. For any m -diagonal α , define the mutation at α to be the new $(m+2)$ -angulation obtained by rotating the $2m+2$ -gon corresponding to α clockwise. See Figure 3.

FIGURE 3. Mutation at the diagonal α .

For an $(m+2)$ -angulation Δ of $P_{n,m}$, we can define a coloured quiver with $n-1$ vertices in the following way (see [BT]). The vertices are the diagonals in Δ and there is an arrow from α to β if they both lie in some $(m+2)$ -gon in Δ . The colour of the arrow is the number of edges forming the segment of the boundary of the $(m+2)$ -gon which lies between α and β , counterclockwise from α . See Proposition 11.1 in [BT]. We write Q_Δ for the coloured quiver obtained from Δ .

It is known that the quiver obtained in this way is a coloured quiver of type A_{n-1} . It is also known that all quivers of this type can be obtained by such an $(m+2)$ -angulation. Also, quiver mutation commutes with mutation at diagonals. Figure 4 shows three 4-angulations of $P_{4,2}$ and the corresponding coloured quivers. The Figure also illustrates mutation at the middle diagonal.

FIGURE 4. Examples of 4-angulations of $P_{4,2}$ and the corresponding quivers.

Let α be an m -diagonal. Denote by $r\alpha$ the m -diagonal obtained by rotating the polygon one step in the counterclockwise direction. Also we let $r\Delta$ be the $(m+2)$ -angulation obtained by applying r on each diagonal in Δ . It is clear that r preserves the corresponding coloured quiver.

Baur and Marsh define in [BM2] a category $\mathcal{C}_{n,m}$ of diagonals. This is a generalization of [CCS], where they consider the case $m=1$. The indecomposable objects are the m -diagonals in this additive category. The morphisms are generated by certain elementary moves of m -diagonals modulo mesh-relations. There is an elementary move $\alpha_{i,j} \rightarrow \alpha_{i,k}$ if both are m -diagonals such that they form

an $(m+2)$ -gon with the boundary, and we can obtain $\alpha_{i,k}$ by rotating $\alpha_{i,j}$ clockwise about the vertex i . Let α be an m -diagonal, and consider all m -diagonals $\beta_1, \beta_2, \dots, \beta_t$ with an elementary move to α , say $f_i : \beta_i \rightarrow \alpha$. Then there exist elementary moves $g_i : r^m \alpha \rightarrow \beta_i$. The mesh-relation at α is defined to be

$$\sum_{i=1}^t f_i g_i.$$

Baur and Marsh show that this category is equivalent to the m -cluster category of Dynkin type A_{n-1} . They also define a quiver associated to the category, where the vertices are the m -diagonals and the arrows are the elementary moves. This quiver is isomorphic to the AR-quiver of the m -cluster category. The AR-translation is denoted by τ , and it is in fact equal to r^m . See Figure 5 for the AR-quiver of $\mathcal{C}_{4,2}$.

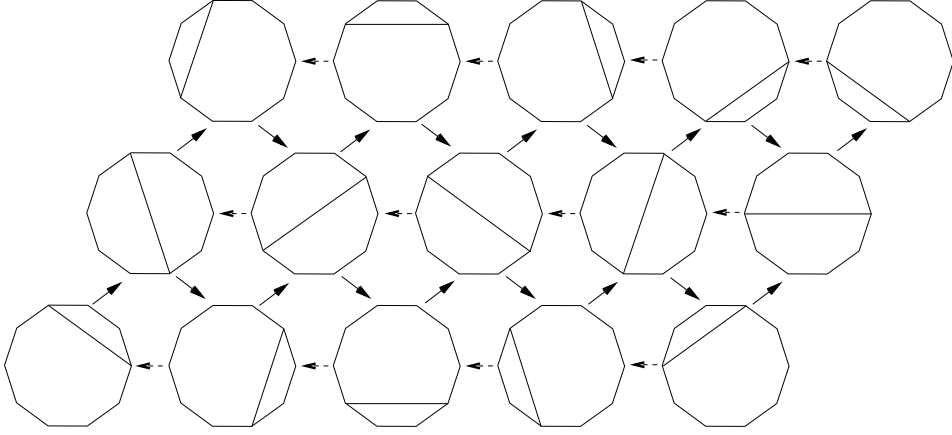


FIGURE 5. AR-quiver of $\mathcal{C}_{4,2}$.

Furthermore, we have that if Δ is an $(m+2)$ -angulation, then Δ , as a direct sum of its m -diagonals, is a basic m -cluster-tilting object. Its endomorphism ring has quiver Q_Δ .

4. BIJECTION BETWEEN THE MUTATION CLASS OF COLOURED QUIVERS OF TYPE A_{n-1} AND m -ANGULATIONS OF POLYGONS

In this section we want to generalize the results in [To1]. The proof of the main theorem uses the same ideas as in [To1]. However, the situation is more complicated, and there are more cases to consider. We prepare for the main theorem with a series of lemmas.

Let $\mathcal{T}_{n,m}$ be the set of $(m+2)$ -angulations of $P_{n,m}$, and denote by $\mathcal{M}_{n,m}$ the mutation class of coloured quivers of type A_n . From the previous section we have a surjective map

$$\sigma_{n,m} : \mathcal{T}_{n,m} \rightarrow \mathcal{M}_{n-1,m},$$

where $\sigma_{n,m}(\Delta) = Q_\Delta$. This function is not injective, since for example $\sigma_{n,m}(\Delta) = \sigma_{n,m}(r\Delta)$.

In the same way as in [To1] we define an equivalence relation on $\mathcal{T}_{n,m}$ by letting $\Delta \sim \Delta'$ if $\Delta' = r^i \Delta$ for some i . Recall that $r^m = \tau$. In [To2] it was observed that if T is an m -cluster tilting object in \mathcal{C}_H^m , then Q_T is isomorphic to $Q_{T[i]}$ for all $i \in \mathbb{Z}$. Since $[i]$ is an equivalence on the m -cluster category, we have that $\text{End}_{\mathcal{C}}(T) \simeq \text{End}_{\mathcal{C}}(T[i])$ for all $i \in \mathbb{Z}$. Considering the category of m -diagonals, we have the

following proposition. The proof is left to the reader, and it is straightforward from the AR-quiver.

Proposition 4.1. *Let $\mathcal{C}_{n,m}$ be the category of m -diagonals of $P_{n,m}$, then $r = [1]$.*

We want to show that the function

$$\tilde{\sigma}_{n,m} : (\mathcal{T}_{n,m}/\sim) \rightarrow \mathcal{M}_{n-1,m},$$

induced from $\sigma_{n,m}$, is bijective. For $m = 1$ this is already known for all n .

We say that an m -diagonal α in an $(m+2)$ -angulation Δ is close to the border in Δ if α lies in an $(m+2)$ -gon together with edges only on the border. If Δ is an $(m+2)$ -angulation and α is close to the border, we define Δ/α to be the $(m+2)$ -angulation of $P_{n-1,m}$ obtained from Δ by letting α be a border edge and leaving all other diagonals unchanged. See Figure 6. We say that we factor out α .

For a diagonal α in some $(m+2)$ -angulation Δ , we always denote by v_α the corresponding vertex in Q_Δ . We generalize the proofs in [To1] to obtain the following lemmas.

Lemma 4.2. *If Δ is an $(m+2)$ -angulation and α close to the border in Δ , then Q_Δ/v_α is connected. Furthermore, factoring out v_α corresponds to factoring out α in Δ , i.e. $Q_{\Delta/\alpha} = Q_\Delta/v_\alpha$.*

Proof. Factoring out α in Δ corresponds to removing the vertex v_α in Q_Δ and all arrows adjacent to v_α . All the other m -diagonals remain unchanged, and hence all arrows between all other vertices remain unchanged. It follows that $Q_{\Delta/\alpha} = Q_\Delta/v_\alpha$. Then it is also easy to see that the resulting quiver is connected, since we obtain a new $(m+2)$ -angulation. \square

Lemma 4.3. *If Δ is an $(m+2)$ -angulation and Q_Δ the corresponding coloured quiver, then the coloured quiver Q_Δ/v_α , obtained from Q_Δ by factoring out a vertex v_α , is connected if and only if the corresponding m -diagonal α is close to the border.*

Proof. If α is not close to the border, α divides Δ into two parts A and B with at least one m -diagonal in each part. Let β be an m -diagonal in A and β' an m -diagonal in B . If β and β' would determine a common $(m+2)$ -gon in Δ , the third m -diagonal would cross α . Hence there is no path between the subquiver determined by A and the subquiver determined by B , except those passing through v_α .

If α is close to the border, then, by Lemma 4.2, factoring out v_α does not disconnect the quiver. \square

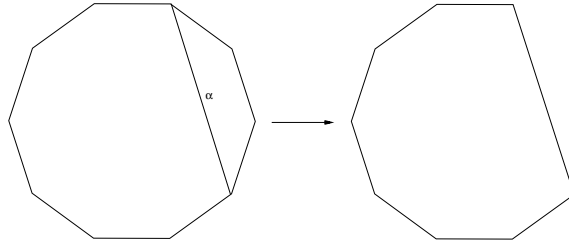


FIGURE 6. Factoring out a diagonal close to the border, where $m = 2$ and $n = 4$.

We have the following proposition.

Proposition 4.4. *Let Q be a coloured quiver of an m -cluster-tilted algebra of type A_{n-1} . If Q/v is the coloured quiver obtained from Q by factoring out a vertex v such that Q/v is connected, then Q/v is a coloured quiver of an m -cluster-tilted algebra of type A_{n-2} .*

Proof. The vertex v corresponds to an m -diagonal α close to the border of an $(m+2)$ -angulation Δ of $P_{n,m}$, by Lemma 4.3. Since Δ/α is an $(m+2)$ -angulation of $P_{n-1,m}$, the claim follows. \square

Next we want to define an extension of an $(m+2)$ -angulation Δ of $P_{n,m}$, by adding m new vertices to the border of the polygon and an m -diagonal α such that α is close to the border and $\Delta \cup \alpha$ is an $(m+2)$ -angulation of $P_{n+1,m}$. More precisely, consider any border edge e between vertex v_i and v_{i+1} on $P_{n,m}$, and extend the polygon with m new vertices between v_i and v_{i+1} , making e the new diagonal close to the border in $P_{n+1,m}$. We denote by $\Delta(e)$ the new $(m+2)$ -angulation thus obtained. See Figure 7. Obviously $\Delta(e)/e = \Delta$. Of course, we can do this at any border edge in Δ .

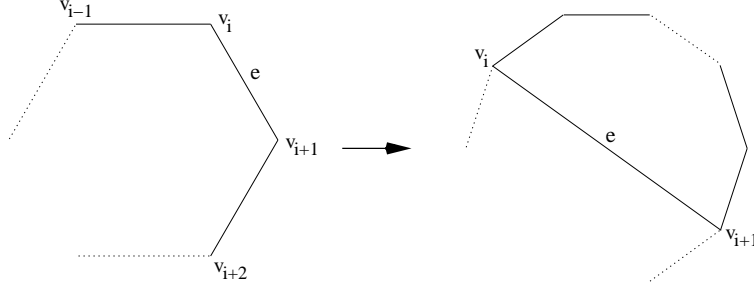


FIGURE 7. Extending an $(m+2)$ -angulation of $P_{n,m}$ at the border edge e , making e a diagonal close to the border in an $(m+2)$ -angulation $\Delta(e)$ of $P_{n+1,m}$.

Let α be some m -diagonal in an $(m+2)$ -angulation Δ . We say that an m -diagonal or border edge e is adjacent to α if e and α both lie in some common $(m+2)$ -gon in Δ . If e is an m -diagonal, then there is some coloured arrow from v_e to v_α in Q_Δ if and only if α and e are adjacent. If e is a border edge of the polygon, then $Q_{\Delta(e)}$ has a coloured arrow from v_e to v_α if and only if e and α are adjacent.

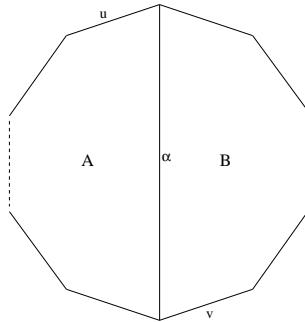


FIGURE 8. Extending an $(m+2)$ -angulation Δ of $P_{n,m}$ such that the vertex corresponding to the new diagonal in $Q_{\Delta(e)}$ is adjacent to the vertex corresponding to α . See Lemma 4.6.

Suppose α is an m -diagonal in an $(m+2)$ -angulation Δ of $P_{n,m}$, and consider Figure 8. Then α divides Δ into two parts A and B . Since there can be at most $(m+1)$ border edges adjacent to α in A and $(m+1)$ border edges adjacent to α in B , there are at most $2(m+1)$ ways to extend Δ such that the vertex corresponding to the new diagonal is adjacent to the vertex corresponding to α . However, by definition of Q_Δ , there is at most one border edge e in A such that $Q_{\Delta(e)}$ has an arrow of a particular colour c from v_e to v_α , and at most one border edge e' in B such that $v_{e'}$ has an arrow of colour c to v_α .

Before we prove the next lemma, recall that $r^i\Delta$ denotes the rotation of Δ i steps in the counterclockwise direction. In the same way we write $r^i\alpha$, where α is an m -diagonal, for the m -diagonal obtained from α by rotating the polygon i steps in the clockwise direction, i.e. $r^i\alpha = \alpha[i]$. We extend this definition to a set of m -diagonals, so let A be a set of diagonals, then we define

$$r^iA = \{r^i\alpha \mid \alpha \in A\}.$$

We denote by Q_{Δ_A} the subquiver of Q_Δ determined by the diagonals only in A , i.e. the subquiver of Q_Δ obtained by factoring out all vertices corresponding to m -diagonals not in A .

Lemma 4.5. *Suppose Δ is an $(m+2)$ -angulation such that $Q_{\Delta'} \simeq Q_\Delta$ for some $(m+2)$ -angulation Δ' implies $\Delta' = r^i\Delta$ for some integer i .*

Let α be an m -diagonal in Δ that divides Δ into two parts A and B . Suppose there is an isomorphism $Q_\Delta \xrightarrow{\theta} Q_{\Delta'}$ that sends v_α to $v_{\alpha'}$. Then α' divides Δ' into two parts A' and B' , such that $\alpha' = r^i\alpha$, $A' = r^iA$ and $B' = r^iB$ for some integer i .

Proof. Suppose there are s m -diagonals in A and t m -diagonals in B .

If $s = t$, the claim is clear, since Δ and Δ' are rotation equivalent, and there are no other m -diagonal in Δ , except α , that divides Δ into two parts with the same number of m -diagonals.

Suppose $s \neq t$. First we make a general definition. Let Δ be an $(m+2)$ -angulation and α an m -diagonal in Δ that divides Δ into two parts A and B . Define $E_{\Delta,\alpha}$ to be the finite set of m -diagonals $\{\alpha_k\}$ in Δ that divide Δ into two parts A_k and B_k such that there exist some integer j with $r^jA_k = A$. The set $E_{\Delta,\alpha}$ is non-empty because $\alpha \in E_{\Delta,\alpha}$. Clearly $Q_{\Delta_{A_k}} \simeq Q_{\Delta_A}$.

Now, consider the situation in the Lemma. Clearly α' divides Δ' into two parts A' and B' such that Q_{Δ_A} is isomorphic to $Q_{\Delta'_{A'}}$ and Q_{Δ_B} is isomorphic to $Q_{\Delta'_{B'}}$. Also, we know that there exist an integer i such that $r^i\alpha = \alpha'$, since both α and α' divides Δ and Δ' into two parts with s and t m -diagonals. Now, suppose for a contradiction that $r^iA \neq A'$.

Construct a new $(m+2)$ -angulation Δ'' by setting $\Delta'' = (r^i\Delta - r^iA) \cup A'$. Then $Q_\Delta \simeq Q_{\Delta''}$, since $Q_{\Delta_{A \cup \alpha}} \simeq Q_{\Delta'_{A' \cup \alpha'}}$. But we have that $|E_{\Delta,\alpha}| > |E_{\Delta'',\alpha}|$ by definition of Δ'' , so Δ is not rotation equivalent to Δ'' . This is a contradiction, so hence $r^iA = A'$. Similarly with B and B' . \square

Lemma 4.6. *Let Δ be an $(m+2)$ -angulation such that if $Q_{\Delta'}$ is isomorphic to Q_Δ for some $(m+2)$ -angulation Δ' , then $\Delta' = r^i\Delta$ for some integer i .*

Let $\Delta(e)$ and $\Delta(e')$, with $e \neq e'$, be two extensions of Δ . Then $\Delta(e) = r^i\Delta(e')$ for some integer i if and only if $Q_{\Delta(e)}$ is isomorphic to $Q_{\Delta(e')}$.

Proof. If $\Delta(e) = r^i\Delta(e')$ for some integer i , then clearly $Q_{\Delta(e)}$ is isomorphic to $Q_{\Delta(e')}$.

Suppose $Q_{\Delta(e)}$ is isomorphic to $Q_{\Delta(e')}$, and suppose that the new m -diagonal e is adjacent to α , say v_e has an arrow of colour c to v_α in $Q_{\Delta(e)}$. The diagonal α

divides Δ into two parts A and B . Suppose there are s m -diagonals in A and t m -diagonals in B . Without loss of generality we can assume that e is a border edge in A .

Since $Q_{\Delta(e)}$ is isomorphic to $Q_{\Delta(e')}$, there exist some v_β in $Q_{\Delta(e')}$ with an arrow of colour c to some vertex v_γ , and such that there exist an isomorphism $Q_{\Delta(e)} \rightarrow Q_{\Delta(e')}$ that sends v_α to v_γ and v_e to v_β . By Lemma 4.3 we know that β is close to the border, since e is close to the border. Then clearly

$$Q_\Delta \simeq Q_{\Delta(e)/e} \simeq Q_{\Delta(e)}/v_e \simeq Q_{\Delta(e')}/v_\beta \simeq Q_{\Delta(e')}/\beta,$$

so hence, by assumption, $\Delta(e')/\beta = r^i \Delta$ for some integer i with $\gamma = r^i \alpha$. We have that γ divides $\Delta(e')/\beta$ into two parts A' and B' , and by Lemma 4.5, we have, say $A' = r^i A$ and $B' = r^i B$.

If β is in A' , then $\beta = r^i e$, and we are finished, since $\Delta(e')/\beta = r^i(\Delta(e)/e)$. If not, then $s = t$, since $B' = r^j A$ for some integer j , and so hence α and γ are both diameters in $P_{n,m}$, i.e. they connect two opposite vertices in the polygon. In any $(m+2)$ -angulation, only one m -diagonal can divide the polygon into two parts with the same number of m -diagonals. But then $j = i + (mn+2)/2$, i.e. $\gamma = r^{i+(mn+2)/2} \alpha$ with $B' = r^{i+(mn+2)/2} A$ and $A' = r^{i+(mn+2)/2} B$.

There is only one way to extend $\Delta(e')/\beta$ in B' such that the new vertex has an arrow of colour c to $v_\gamma = v_\alpha$, so $\beta = r^{i+(mn+2)/2} e$. Consequently $r^{i+(mn+2)/2} \Delta(e) = \Delta(e')$, and we are done. \square

Now we can prove the following theorem by induction.

Theorem 4.7. *The function*

$$\tilde{\sigma}_{n,m} : (\mathcal{T}_{n,m}/\sim) \rightarrow \mathcal{M}_{n-1,m}$$

is bijective for all n and m .

Proof. We already know that the function is surjective. Let m be a positive integer and suppose $\tilde{\sigma}_{n,m}(\Delta) = \tilde{\sigma}_{n,m}(\Delta')$. We want to show that $\Delta = \Delta'$ in $(\mathcal{T}_{n,m}/\sim)$. It is straightforward to check that

$$\tilde{\sigma}_{3,m} : (\mathcal{T}_{3,m}/\sim) \rightarrow \mathcal{M}_{2,m}$$

is injective for all m .

Suppose that

$$\tilde{\sigma}_{n-1,m} : (\mathcal{T}_{n-1,m}/\sim) \rightarrow \mathcal{M}_{n-2,m}$$

is injective for all m . Let α be an m -diagonal close to the border in Δ with image v_α in Q , where Q is a representative for $\tilde{\sigma}_{n,m}(\Delta)$. Then the diagonal α' in Δ' corresponding to v_α in Q is also close to the border. We have that $\tilde{\sigma}_{n-1,m}(\Delta) = \tilde{\sigma}_{n-1,m}(\Delta')$, and so by hypothesis $\Delta/\alpha = \Delta'/\alpha'$ in $(\mathcal{T}_{n,m}/\sim)$.

We can obtain Δ and Δ' from Δ/α and Δ'/α' by extending the polygon at some border edge. By Lemma 4.6 all possible extensions of Δ/α and Δ'/α' gives non-isomorphic quivers, unless $\Delta = \Delta'$ in $(\mathcal{T}_{n,m}/\sim)$, and that finishes the proof. \square

Corollary 4.8. *The number of elements in the m -mutation class of coloured quivers of type A_{n-1} is the number of $(m+2)$ -angulations of $P_{n,m}$, where two $(m+2)$ -angulations Δ and Δ' are equivalent if Δ' can be obtained from Δ by rotation.*

We see that this agrees with the results obtained in [To1] for $m = 1$, where the number of quivers in the mutation class of A_{n-1} was shown to be the number of triangulations of the disk with n diagonals, i.e. the number of triangulations up to rotation of a regular polygon with $n+2$ vertices.

5. THE NUMBER OF QUIVERS IN THE m -MUTATION CLASS OF A AND THE CELL-GROWTH PROBLEM

In this section we want to give formulas for the number of quivers in the m -mutation classes of coloured quivers of Dynkin type A .

It is easy to compute the number of non-isomorphic indecomposable objects in \mathcal{C}_H^m , and we just give the formula without proof. The number of indecomposable objects in \mathcal{C}_H^m of Dynkin type A_n is given by

$$\frac{mn(n+1) + 2n}{2}.$$

It is known from [Z] that there is a bijection between the set of m -clusters as defined in [FR] and the set of m -cluster tilting objects. The m -cluster tilting objects are in 1 – 1 correspondence with $(m+2)$ -angulations, but as we have seen two tilting objects may give rise to the same cluster-tilted algebra and the same coloured quiver. The number of m -cluster tilting objects is the number of $(m+2)$ -angulations, where two $(m+2)$ -angulations are considered distinct if they are rotations of each other. In the case $m=1$ this is the number of triangulations of a polygon, and is known to be the Catalan numbers. This also agrees with [FZ2], which says that the number of clusters of Dynkin type A is the Catalan numbers.

The number of m -clusters of type A is known from [FR]. These numbers are called the Fuss-Catalan numbers of type A . Fuss proved already in 1791 that the number of $(m+2)$ -angulations of an $(mn+2)$ -gon is given by these numbers [F]. See [A] for a historical reference and more details.

The number of $(m+2)$ -angulations of a polygon is related to the cell-growth problem, which has been investigated by many authors. We will use the results obtained in [HPR] by Harary, Palmer and Read in 1975.

The cell-growth problem considers an “animal” that grows from a cell with a certain shape, i.e number of sides. See Figure 9 for some examples of such “animals”. The problem considered in [HPR] is to count the number of such “animals” with a fixed number of cells. They also consider “animals” where two are equivalent if one can be brought into the other by rotations and reflections.

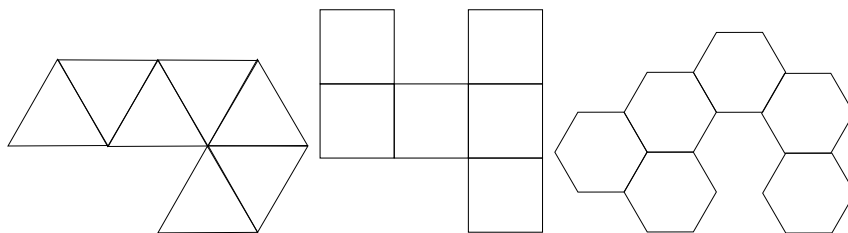


FIGURE 9. Examples of “animals” with cells the shapes of polygons with 3,4, and 6 vertices.

In [HPR] they consider “animals” that have a certain tree-like structure. We refer to the paper for details. We can draw a node inside each cell and an edge between two nodes if and only if they are adjacent. This should result in a connected tree and not a graph. See Figure 10.

The cells have the shape of a polygon with s vertices, and an “animal” with allowed tree-like structure is called an s -cluster. This is a rather fortunate name, because we will see that such clusters or “animals” are in 1 – 1 correspondence with cluster-tilting objects, which again are in 1 – 1 correspondence with clusters in the cluster algebra sense. We will try not to confuse the two different meanings in the rest of this paper.

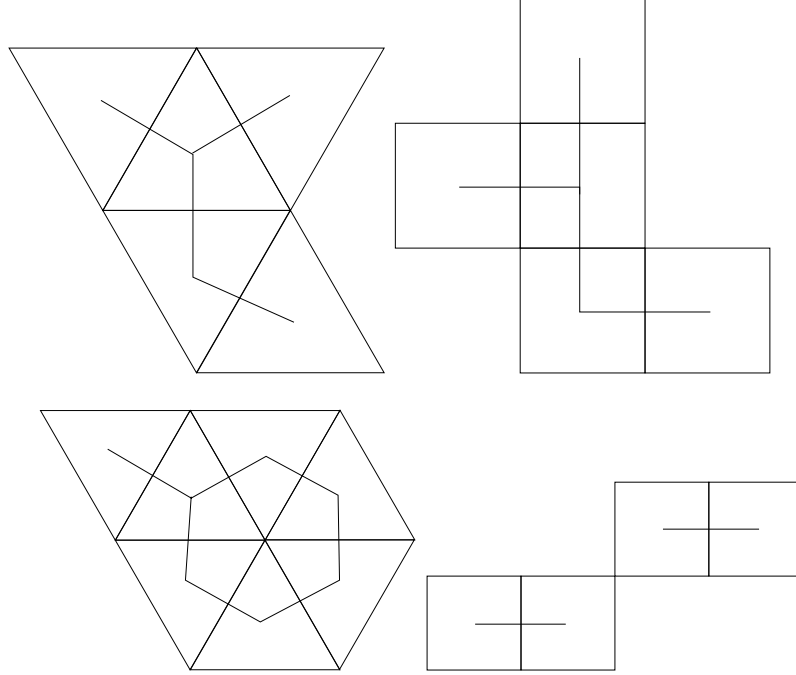


FIGURE 10. The two first pictures are “animals” with allowed tree-like structure, while the third and fourth are not allowed.

It is well-known that counting the number of s -clusters is the same as counting the number of s -angulations of polygons. This follows because of the tree-structure of the clusters. Given an s -angulation of a polygon, we can construct an s -cluster, and given an s -cluster we can construct an s -angulation. See Figure 11 for some examples. To construct a cluster, we dissect the polygon into s -gons given by its s -angulation, and we let two cells be adjacent if they were adjacent in the polygon. To construct an s -angulation, we simply let the outer edges of the cluster be the border of the polygon.

First we consider the number of $(m+2)$ -angulations of $P_{n,m}$, where we fix the polygon in the plane and consider two $(m+2)$ -angulations to be distinct if one can be rotated into the other. We use the same notation as in [HPR]. Let

$$U_s(x) = \sum_{r=1}^{\infty} U_r^{(s)} x^r,$$

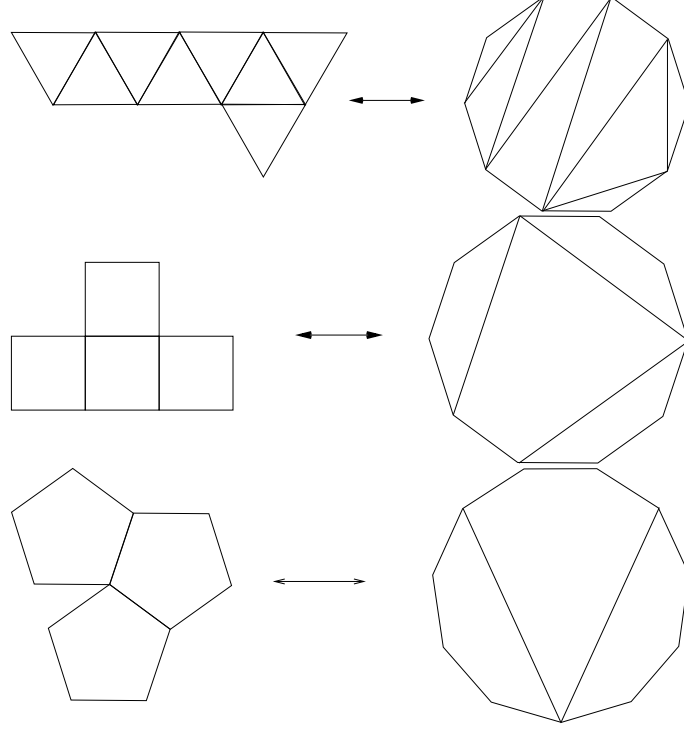
where $U_r^{(s)}$ denotes the number of s -clusters of r cells, rooted at an outer edge. We have the following equality [HPR, P].

$$U_s(x) = x(1 + U_s(x))^{n-1}$$

By an application of Lagrange’s inversion theorem they obtain that

$$U_r^{(s)} = \frac{1}{r} \binom{r(s-1)}{r-1}.$$

By the above this also gives the number of s -angulations of a polygon with $r(s-2)+2$ vertices.

FIGURE 11. Correspondence between s -clusters and s -angulations.

We obtain the following directly, and these are the Fuss-Catalan numbers mentioned in the beginning of this section. Note that this is a direct consequence of the correspondence in [Z] in combination with the formula of [FR], see also [A].

Theorem 5.1 ([Z, FR]). *The number of m -cluster tilting objects in the m -cluster category \mathcal{C}_H^m of type A_n is given by*

$$U_{n+1}^{(m+2)} = \frac{1}{n+1} \binom{(n+1)(m+1)}{n}.$$

Proof. This follows from the above discussion and the fact that $(m+2)$ -angulations are in 1 – 1 correspondence with m -cluster tilting objects. \square

Observe that these numbers give the Catalan numbers when $m = 1$.

Next, we want to count the number of coloured quivers in the m -mutation class, and by Corollary 4.8 we consider $(m+2)$ -angulations that are equivalent under rotation. In [HPR] they consider s -clusters rooted at a cell, and by applying Pölyas Theorem they obtain the series

$$F_s(x) = \sum_{r=1}^{\infty} F_r^{(s)} x^r = xZ(C_s; 1 + U_s(x)),$$

where F_r^s gives the number of s -clusters with r cells rooted at a cell and C_s is the cyclic group of order s .

Next they consider unrooted clusters, and by applying Pölyas Theorem again they obtain give the generating function

$$H_s(x) = F_s(x) - \frac{1}{2}(U_s^2(x) - U_s(x^2)).$$

This counts the number of unrooted s -clusters, and so hence s -angulations of polygons, where two s -angulations are equivalent if they can be rotated into one another.

Now we want to compute the coefficients to obtain explicit formulas, which they do not do in [HPR].

First, we consider $U_s^t(x)$ for some t . We apply Lagrange's inversion theorem (see for example [HP], Chapter 1.7) on the equation

$$U_s(x) = x(1 + U_s(x))^{n-1}.$$

Set $y = U_s(x)$, and we get

$$y^t = \sum_{i=1}^{\infty} \frac{x^i}{i!} \left(\left(\frac{d}{dy} \right)^{i-1} \left(y^{t-1} (1+y)^{(n-1)i} \right) \right)_{y=0}.$$

Differentiating and substituting we get

$$U_s^t(x) = \sum_{i=t}^{\infty} \frac{t}{i} \binom{i(s-1)}{i-t} x^i.$$

We see that this agrees with the expression for $U_s(x)$ for $t = 1$.

Now we compute $F_s(x)$. We get

$$\begin{aligned} F_s(x) &= xZ(C_s; 1 + U_s(x)) = \\ &= x \frac{1}{s} \sum_{d|s} \phi(d) (1 + U_s(x^d))^{s/d} = \\ &= \frac{1}{s} \sum_{d|s} \phi(d) x \sum_{t=0}^{s/d} \binom{n/d}{t} U_s(x^d)^t = \\ &= \frac{1}{s} \sum_{d|s} \phi(d) x \sum_{t=0}^{s/d} \binom{n/d}{t} \sum_{i=t}^{\infty} \frac{t}{i} \binom{i(s-1)}{i-t} x^{di} = \\ &= \frac{1}{s} \sum_{d|s} \phi(d) x \left(1 + \sum_{t=1}^{s/d} \binom{n/d}{t} \sum_{i=t}^{\infty} \frac{t}{i} \binom{i(s-1)}{i-t} x^{di} \right) = \\ &= \frac{1}{s} \sum_{d|s} \phi(d) \left(x + \sum_{t=1}^{s/d} \binom{n/d}{t} \sum_{i=t}^{\infty} \frac{t}{i} \binom{i(s-1)}{i-t} x^{di+1} \right) \\ &= \frac{1}{s} \sum_{d|s} \phi(d) x + \frac{1}{s} \sum_{d|s} \phi(d) \sum_{t=1}^{s/d} \binom{n/d}{t} \sum_{i=t}^{\infty} \frac{t}{i} \binom{i(s-1)}{i-t} x^{di+1} = \\ &= x + \frac{1}{s} \sum_{d|s} \sum_{t=1}^{s/d} \sum_{i=t}^{\infty} \phi(d) \frac{t}{i} \binom{n/d}{t} \binom{i(s-1)}{i-t} x^{di+1}. \end{aligned}$$

We are interested in the coefficients of this series. Clearly the coefficient of x is always 1. Suppose we want the coefficient of x^k . Then $di + 1 = k$ and so $i = \frac{k-1}{d}$. This means that we want to sum over all d that also divides $k-1$. Also we want to sum over all t which give a contribution, that is up to $\min\{\frac{s}{d}, \frac{k-1}{d}\}$. We get that the coefficient of x^k is given by

$$= \sum_{d|s \text{ \& } d|(k-1)} \sum_{t=1}^{\min\{\frac{s}{d}, \frac{k-1}{d}\}} \left[\frac{\phi(d)dt}{s(k-1)} \binom{s/d}{t} \binom{(s-1)\frac{k-1}{d}}{\frac{k-1}{d} - t} \right].$$

This formula counts the number of clusters rooted at a cell.

Now we need to compute the coefficients of

$$\frac{1}{2}(U_s^2(x) - U_s(x^2)).$$

We get

$$\sum_{i=2}^{\infty} \frac{1}{i} \binom{(s-1)i}{i-2} x^i - \sum_{j=1}^{\infty} \frac{1}{2j} \binom{(s-1)j}{j-1} x^{2j}.$$

The coefficient of x^k is given by

$$\frac{1}{k} \binom{(s-1)k}{k-2} - \frac{1}{k} \binom{(s-1)(k/2)}{(k/2)-1},$$

where the last term is omitted if $2 \nmid k$.

We obtain the following theorem, which follows from the above discussion and Corollary 4.8. See some examples in Figure 12.

Theorem 5.2. *The number of non-isomorphic coloured quivers in the m -mutation class of a quiver of Dynkin type A_n is given by*

$$\sum_{d|n \text{ \& } d|m+2} \sum_{t=1}^{\min\{\frac{m+2}{d}, \frac{n}{d}\}} \left[\frac{\phi(d)dt}{(m+2)n} \binom{(m+2)/d}{t} \binom{(m+1)\frac{n}{d}}{\frac{n}{d} - t} \right] - \frac{1}{n+1} \binom{(m+1)(n+1)}{n-1} + \frac{1}{n+1} \binom{(m+1)(n+1)/2}{(n+1)/2-1},$$

where the last term is omitted if $2 \nmid n+1$.

From this formula we can obtain the formula given in [To1] for the number of elements in the mutation class of A_n . In this case $m = 1$, and d takes the value 1 and the value 3 if $3|n$. After some calculations we obtain the formula, namely

$$C(n+1)/(n+3) + C((n+1)/2)/2 + (2/3)C(n/3),$$

where $C(i)$ is the i 'th Catalan number and the second term is omitted if $(n+1)/2$ is not an integer and the third term is omitted if $n/3$ is not an integer.

6. FURTHER REMARKS

In [M] the author gives a description of the Gabriel quivers in the m -mutation class of quivers of Dynkin type A together with the relations. Let Q be a quiver in the m -mutation class of A_n , and let Δ be a corresponding $(m+2)$ -angulation. Then

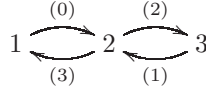
$$i \rightarrow j \rightarrow k$$

is a zero path if and only if the corresponding m diagonals of i , j and k are in the same $(m+2)$ -gon in Δ . It is also shown that there are no commutativity relations.

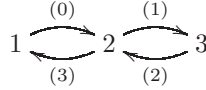
Let Δ and Δ' be two $(m+2)$ -angulations, and Q_Δ and $Q_{\Delta'}$ the corresponding coloured quivers. We know that Q_Δ and $Q_{\Delta'}$ are isomorphic if and only if Δ and Δ' are rotations of each other. However, if Q_Δ and $Q_{\Delta'}$ are not isomorphic as coloured quivers, the Gabriel quivers may still be isomorphic. Also, two non-isomorphic coloured quivers can give rise to the same m -cluster tilted algebra. For example, if Q is the quiver

	m=1	m=2	m=3	m=4
n=2	1	2	2	3
n=3	4	7	12	19
n=4	6	25	57	118
n=5	19	108	366	931
n=6	49	492	2340	7756
n=7	150	2431	16252	68685
n=8	442	12371	115940	630465
n=9	1424	65169	854981	5966610
n=10	4522	350792	6444826	57805410
n=11	14924	1926372	49554420	571178751
n=12	49536	10744924	387203390	5737638778
n=13	167367	60762760	3068067060	58455577800
n=14	570285	347653944	24604111560	602859152496
n=15	1965058	2009690895	199398960212	6283968796705
n=16	6823410	11723100775	1631041938108	66119469155523
n=17	23884366	68937782355	13451978877748	701526880303315
n=18	84155478	408323229930	111765327780200	7498841128986110
n=19	298377508	2434289046255	934774244822704	80696081185767000
n=20	1063750740	14598011263089	7865200653146910	873654669882575000

FIGURE 12. Some numbers of coloured quivers in the mutation classes for various values of n and m .



occurring for a 3-cluster tilted algebra, then the quiver Q'



obtained from Q by mutating at 3, is not isomorphic to Q as coloured quivers. However, they give the same 3-cluster tilted algebra.

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